duration and severity than would be encountered within the Venus atmosphere.

3) The amplitude of lateral and longitudinal oscillations of a tethered balloon is a direct function of the volume of gas within the balloon; the balloon oscillations appear to be less pronounced while inflation is taking place as contrasted to those experienced in a steady-state condition of specific volumes without gas entering the balloon.

4) The longitudinal oscillations of the balloon have a frequency of about $\frac{1}{2}$ cps at maximum amplitude. The lateral oscillations include a rolling moment, and with the tether taut, the balloon end cap describes a circular motion about the longitudinal centerline of the balloon.

For additional development the initial component developments and system feasibility demonstrations described herein should be followed by additional system functional tests, a comprehensive materials study for selection of an appropriate sterilizable balloon material, and a flight demonstration of the complete BVS under simulated conditions in the Earth's atmosphere.

References

[1] Carrol, P. C., Frank, R. E., and Pettus, J. D., "Buoyant Venus Station Mission Feasibility Study for 1972–1973 Launch Opportunities," NASA CR-66725-3, Jan. 1969, Martin Marietta Corp., Denver, Col.

[2] Steinberg, S. and Holle, G. F., "Experimental Results of a Diffuser Configuration Designed for Rapid Inflation of a Superpressure Balloon," FTD Report 0473-68-01, June 1968, Martin Marietta Corp., Denver, Colo.

[3] Holle, G. F., Steinberg, S., and Carroll, P. L., "Wind Tunnel Test Results of the Deployment/Inflation Characteristics of a Model Balloon (Buoyant Venus Station Concept)," MCR-69-23, Jan. 1969, Martin Marietta Corp., Denver, Colo.

[4] *Royal Aeronautical Society Data Sheets*, Vol. 5, Issue 15, No. 02.00.01–02.00.02, 1964.

[5] Steinberg, S., "Pretest Report and Test Plan for a Preliminary BVS Balloon Wind Tunnel Test," MCR-68-456, Nov. 1968, Martin Marietta Corp., Denver, Colo.

JULY 1970 J. SPACECRAFT VOL. 7, NO. 7

Longitudinal Temperature Profiles in Fluid Flow and Considerations in Thermal Modelling

THEODORE C. YORK*
Toronto, Canada

A method for the inclusion of fluid flow phenomena in a heat-transfer computer program has been developed that relieves the difficulties presently experienced with stringent stability criteria. These difficulties are because of the assumption that T_{av}, the average between inlet and outlet temperatures, can always be used as bulk temperature (T_b). This assumption has been made to remedy the condition of two unknowns, T_b and the outlet temperature (T_0), whereas there is only one equation for the heat balance in a fluid nodal volume. The change in T_0 can always be derived from heat-transfer relationships, thus eliminating one of the two unknowns. The other unknown, T_b, can be computed correctly from the energy balance. Hence the various components that contribute to the energy balance in a fluid volume are separated and can be dealt with individually, whereas previously they were obscured by the assumption that $T_b = T_{av}$. Only when there is a full gradient through the node does $T_b = T_{av}$. Some restrictions that still exist because of the linear assumptions of the method are pointed out.

Nomenclature†

CAP = thermal capacitance of fluid node, Btu/°F Fortran notation is used for clarity
c_p = specific heat of fluid, Btu/(°F − lb)
h_c = convective heat transfer coefficient, Btu/(hr − °F − ft^2)
L = length of nodal volume, ft
P = perimeter of enclosure, ft
$\dot{q}$ = rate of heat flow to fluid, Btu/hr
Δt_H = hydraulic time step defined in text
Δt_t = time step dictated by thermal system
T = temperature, °F; $T_{av} = (T_i + T_0)/2$
ΔT = $T_i - T_0$
$\dot{w}$ = flow rate of fluid, lb/hr

Subscripts

b = bulk
i,o = inlet and outlet of node, respectively
w = at wall

Presented as Paper 69–30 at the AIAA 7th Aerospace Sciences Meeting, New York, January 20–22, 1969; submitted April 3, 1969; revision received February 11, 1970. Most of the work described in this paper was performed during employment with McDonnell Douglas Astronautics Company, Huntington Beach, Calif. The author would like to thank W. P. Sokeland, also formerly with McDonnell Douglas Astronautics Company, for his suggestions and contributions in support of this work.

* Consulting Engineer.

† Unprimed quantities are the properties at the beginning of a time step. Primed quantities are the forward properties at the end of a time step.

Introduction

CLASSICAL textbooks on heat transfer, while exploring the evaluation of convective heat-transfer coefficients to great depth, give little information on how to evaluate temperatures in the fluid numerically. The advent of computers led to the advancement of numerical techniques. Several methods were developed for evaluating the Fourier and the Laplace equations. They are used in the many heat-transfer programs existing throughout the industry and considerable literature is available describing them.[1–3] Hence these methods are assumed known and are not dealt with in this paper.

This paper deals with the inclusion of fluid flow phenomena in an existing heat-transfer program. This program uses the method of finite differences, that is, all parameters are assumed known at the beginning of a time step, heat fluxes, conductances, and capacitance values are assumed constant through the time step, and the changes in heat balances and temperatures during the time step are computed explicitly. The forward temperatures are the beginning values for the next time step. This approach constitutes a linearization of the Fourier equation, and the method dictates a critical thermal time step (Δt_t) to preserve stability of the computations.

Inclusion of fluid flow phenomena introduced additional stability criteria. Criteria associated with early numerical techniques for the evaluation of the energy balance in a fluid volume were severe and difficult to observe during modelling. The method described in this paper greatly relieves these difficulties. In accord with the numerical techniques used by the parent heat-transfer program, a forward differencing scheme is employed, and linearity of gradients and slug flow as flow characteristics are assumed.

Today much is known and available in modern texts on temperature distribution through the fluid, flow characteristics, mixing, etc. Such refinements are not included in the method presented here. However, the fact that all components that contribute to the energy balance in a fluid volume are now dealt with individually, permits the introduction of more refined theory should a need for greater accuracy bring about this requirement.

The program uses the Newton equation

$$\dot{q} = PLh_c(T_w - T_b)$$

to compute the heat-transfer rate from the wall. To simplify the equations in the paper, $\dot{q}$ is used to represent this heat flow and is assumed to be computed in this manner. All equations in the text are stated in finite difference form.

Problem

To demonstrate the problem, let us consider a long tube that conducts fluid from a reservoir maintained at constant temperature. The fluid is noncompressible and has considerable thermal capacitance. A heating element supplies uniform heat flow for the entire length of the tube. The thermal capacitance of the tube material is negligible. Therefore, the temperature of the fluid rises uniformly with time over the full length of the tube and

$$T_b' - T_b = \dot{q}(\Delta t/\mathrm{CAP}) \tag{1}$$

represents the change in bulk temperature during a time interval Δt in each nodal volume of capacitance CAP.

A special condition exists at the inlet. The particle that has entered the tube at the beginning of Δt is now downstream a distance L, and its temperature has risen to T'. The particle that arrives at the end of the time step enters with the temperature of the reservoir. Hence, a gradient exists over length L. If a nodal volume had exactly this length, then its bulk temperature would be T_{av}.

A difficulty arises in computing the change of energy in the volume during the time step, because there are two unknowns, the changes in outlet temperature T_0 and bulk temperature T_b. Reference 1 overcomes this difficulty by setting $T_b = T_{av} = (T_i + T_0)/2$. In the example given, this assumption is correct when the length of the fluid node equals L. The inaccuracies introduced when the length does not equal L are explored later, and the implications on modelling are summarized.

For this reason, the following approach was adopted: since T_b is a measure of the energy stored in a nodal volume, not necessarily equal to T_{av}, it can be computed from the heat balance if the change in T_0 can be obtained by different means. The following sections show that the change in T_o can be obtained rationally from heat-transfer relationships. Three driving parameters are explored, separately and then in combinations: heat transferred from the wall, enthalpy in the stream, and change in enthalpy transport.

While the finite-difference formulation of the heat balance equation assumes all parameters constant over the time step, it is now necessary to include the change in T_0, that is, the change in enthalpy leaving the nodal volume. In other words, it is necessary to add the second derivative into the heat balance equation for the fluid node.

Inlet temperatures and their changes are assumed known. In the reservoir, temperature histories must be given or be calculable. For any node in the tube, T_i is equal to the T_0 of the upstream node which has just been computed.

Wall as Driver

In assigning values to the foregoing example, the intent is to be illustrative. (A realistic case is demonstrated later.) Let $\dot{w} = 2$ lb/hr, $c_p = 1$, Btu/lb-°F, and mass in the volume = 5 lb; hence, CAP = 5 Btu/°F. Let $\dot{q} = 50$ Btu/hr, and the thermal network dictates a time step $\Delta t_t = 1$ hr. All temperatures are initially 100°F.

The time necessary for the entire mass in the volume to be replaced is $\frac{5}{2} = 2.5$ hr. Hence the thermal time step governs; $\Delta t = 1$, and the fluid moves during this time $1/2.5 = 0.4$ L. Except at the inlet, the uniformly distributed heating element causes a temperature rise so that Eq. (1) gives $T_b' = 100 + \frac{1}{5}(50) = 110$°F. Referring to Fig. 1, at the end of the first time step there is a gradient from 100° to 110°F over 0.4 L, and in the remainder of the node the temperature of the fluid is 110°F. The mean temperature in the nodal volume, computed from the geometry of the profile, is $0.4 \times 105 + 0.6 \times 110 = 108$°F.

The outlet temperature of the node has risen by 10°F. The enthalpy carried away by the stream during Δt, due to change in T_0 is $\frac{1}{2}\,\dot{w}c_p\Delta t(dT_0/dt)\Delta t = 10$ Btu.

Thus, writing the equation for the energy balance in the volume and including the change in outlet condition, the new bulk temperature is

$$T_b' = T_b + [\dot{q} + \dot{w}c_p\Delta T - \tfrac{1}{2}\dot{w}c_p(dT_0/dt)\Delta t]\Delta t/\mathrm{CAP} \tag{2}$$

Substitution of values yields a T_b' of 108°F, the same as was obtained above by geometry. This is the temperature that must be used in the Newton equation for heat transfer with the wall. Thus, T_b is a measure of the heat in a nodal volume and is not necessarily the actual temperature at the

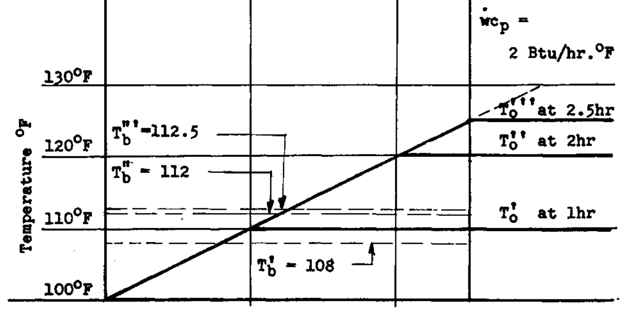

Fig. 1 Development of gradients when system is driven by heat transfer from wall.

midpoint of the node. In Fig. 1, the temperature of the stream at the midpoint at the end of the first time step is 110°F. The outlet temperature is an actual temperature.

If T_{av} were to be used for T_b, the length of the node would have to be restricted to 0.4 L. Computing T_b by the energy balance for the volume removes this restriction on L, and the degree of accuracy of the values obtained is the same as is obtained for the over-all thermal network.

Assuming all input values to remain the same over a second time step Δt = 1 hr, the rate of change of T_0 is still $dT_0/dt = \frac{50}{5} = 10$°F/hr, and substituting into Eq. (2) yields $T_b'' = 112$°F; $T_0'' = 120$°F.

An attempt to repeat the same procedure over a third time step Δt = 1 hr would yield T_0''' = 130°F and, from Eq. (2), T_b''' = 112°F. This of course cannot be. That $T_b'' = T_b'''$ would mean that no change in energy storage took place, yet T_0 went up by 10°F.

Examining this condition more closely shows the existence of a stability criterion. $\dot{q}$ = 50 Btu/hr has remained constant throughout the problem. At the end of the first time step ΔT = −10°F and the heat being carried away in the stream at that instant is $\dot{w}c_p\Delta T$ = − 20 Btu/hr. At the end of the second time step ΔT = −20°F and $\dot{w}c_p\Delta T$ = −40 Btu/hr. Performing this procedure over a third time step would yield ΔT = −30°F and $\dot{w}c_p\Delta T$ = −60 Btu/hr. The heat carried away in the stream would have become greater than $\dot{q}$, which would violate the first law. When the heat carried away in the stream becomes equal to $\dot{q}$, steady state is reached, and T_0 ceases to rise. In the above example steady state is reached when $\dot{w}c_p\Delta T$ = −50 Btu/hr. A ΔT greater than 25°F is therefore not possible.

Stating this in more general terms; by definition of the finite difference formulation, there exists a linear gradient from inlet to outlet through a node, when it is at steady state and $T_b = T_{av}$. If this steady state is upset by a change in $\dot{q}$, the system will strive toward a new steady-state condition that will be in equilibrium with the new $\dot{q}$, and T_0 will change, as computed in the above example. It will take Δt_H hr to reach the new steady state, at which time $T_b' = T_{av}' = T_{av} + \frac{1}{2}(dT_0/dt)\Delta t_H$. Substituting this into Eq. (2) and rearranging the terms yields

$$(T_b - T_{av})\mathrm{CAP} + [\dot{q} + \dot{w}c_p\Delta T - (\mathrm{CAP}/2)dT_0/dt]\Delta t_H - \tfrac{1}{2}\dot{w}c_p(dT_0/dt)\Delta t_H^2 = 0 \quad \text{(2a)}$$

Solving this equation yields the hydraulic time step Δt_H, i.e., the maximum Δt that can be used in Eq. (2). When the system starts from a steady-state condition the capacitance term will be zero. If it does not, the amount of energy stored in the volume will greatly affect Δt_H.

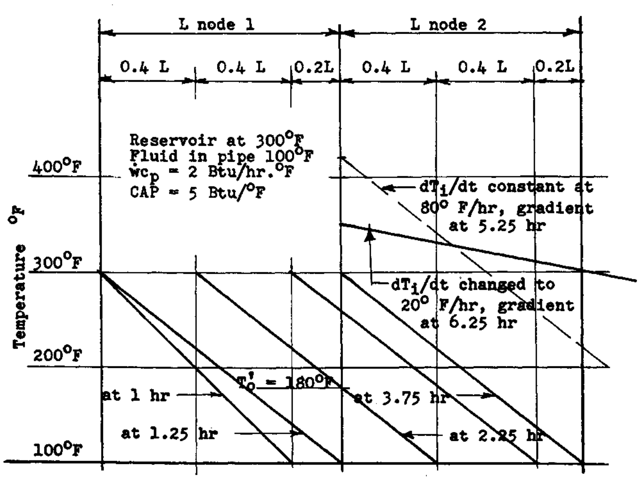

Fig. 2 Development of gradient when system is driven by enthalpy in stream.

Returning to our example, temperatures had been calculated at the end of the second time step. Entering these values into Eq. (2a) a critical Δt_H = 0.5 hr is obtained. Using this time step, $T_0''' = T_0'' + (dT_0/dt)\Delta t$ = 125°F, and from Eq. (2) T_b''' = 112.5°F. At this point steady state is reached.

Hydraulic Time Step Governs

In the foregoing example it was assumed that the thermal network dictated a time step Δt_t = 1 hr. Three such time steps were used. During the third step a new criterion came up, Δt_H, the hydraulic time step of 0.5 hr. At the end of this step the node was at steady state, that is $T_b = T_{av}$ and T_0 stopped rising.

However, there remained 0.5 hr to complete Δt_t of 1 hr. Since the node has reached steady state no further changes take place during this interval and no computation can be performed.

The foregoing example was started from an isothermal condition of 100°F. If at the beginning of the example Δt_t had been assumed to be 4 hr instead of 1 hr, Eq. (2a) would have revealed a critical Δt_H of 2.5 hr. At the end of this interval T_0' = 125°F and, from Eq. (2), T_b' = 112.5°F. These values remain unchanged during the remaining 1.5 hr of Δt_t. An attempt to let T_0 rise through the entire 4-hr interval would result in oscillations with erroneous temperatures.

Enthalpy in Stream as Driver

Now let us assume the same system, except that the tube of low capacitance is insulated, so that no heat can leave the system. Fluid in the tube is at 100°F. The reservoir is maintained at 300°F and is separated from the tube by a gate. Suddenly the gate is released, and flow takes place at 2 lb/hr. Again, Δt_t = 1 hr, and the length of flow during this time is 0.4 L. From consideration of mass balance at the end of the first time step, $\frac{4}{10}$ of the volume is filled by fluid of 300°F, $\frac{6}{10}$ by fluid of 100°F, resulting in an average temperature of 180°F.

Substitution into Eq. (2) yields the same result, $T_b = 100 + \frac{1}{5}[2(200)]$ = 180°F. Referring to Fig. 2, there is a linear gradient from 300° to 100°F. But since the resulting T_b is less than T_{av}, the only way Eq. (2) can be satisfied is if the gradient stops short of the end of the node as is shown in Fig. 2. This means, no change in T_0 has taken place during this interval [dT_0/dt had been considered to be zero in above evaluation of Eq. (2)]. Substitution of the same beginning values into Eq. (2a) gives a Δt_H of 1.25 hr, which is the interval required to develop the full gradient through the node. This can readily be verified by considering the mass balance. At this point in time $T_b = T_{av}$ = 200°F, and up to this point no change in T_0 has taken place. However, since the node is not in thermal equilibrium, the gradient is not stable. From here on T_0 must rise.

Ultimately the entire node will be filled by fluid of 300°F, i.e., T_b' = 300°F, and by combining Eqs. (2) and (2a), dT_0/dt and Δt_H can be obtained. However, this derivation is deferred until later.

Until now the fluid, which entered at 300°F, was assumed cooled by mixing. Linearity of mixing was assumed resulting in linear gradients. Once the full gradient is reached, the point where the mixing starts has to move inward into the node. The particle which now enters, remains at 300°F and reaches the outlet of the node in $\frac{5}{2}$ = 2.5 hr, at which time the entire node is filled with liquid of that temperature. Thus T_0 rises during this time to 300°F, and dT_0/dt = 200/2.5 = 80°F/hr.

One hour after the full gradient was reached (i.e., 2.25 hr after the opening of the gate) T_0' = 180°F and, from Eq. (2), T_b' = 264°F. During this time the particle which has remained at 300°F has moved 0.4 L into the node, that is, the gradient has shifted downstream, as can be seen in Fig. 2. As before, $T_b' = 264$°F can be verified by geometry, since the area under the gradient represents the energy in the volume. Conversely, if T_b and T_0 are known at the beginning of a time step, geometry can be used to locate the gradient.

Using the values just obtained as beginning values for a new time step, Eq. (2a) reveals $\Delta t_H = 1.5$ hr. Since T_0 must rise during this interval from 180° to 300°F, $dT_0/dt = 120/1.5 = 80$°F/hr, and substitution into Eq. (2), using $\Delta t = 1.5$, yields $T_b'' = 300$°F; $T_0'' = 300$°F.

This discussion shows that enthalpy brought in by the stream is first used to develop the full gradient, during which time T_0 does not change. Once the full gradient is developed, T_b starts to rise above T_{av}, and the energy stored in the volume forces T_0 to rise until all the fluid in the node has the temperature of the inlet.

If $\Delta t_t = 4$ hr had been used in this example, this time step would have had to be subdivided twice. During the first 1.25 hr, $dT_0/dt = 0$, and during the next 2.5 hr, $dT_0/dt = 80$°F/hr. At 3.75 hr steady state is reached, the enthalpy carried in equals the enthalpy carried out, and no change can take place during the remaining 0.25 hr.

A rise in T_0 also constitutes a rise in T_i of the next node downstream, and therefore, a change in the enthalpy carried in by the stream. In fact, temperatures in all downstream nodes are changed by these changes in enthalpy transport.

Change in Enthalpy Transport as Driver

It now becomes necessary to introduce dT_i/dt, which causes a change in enthalpy in the volume of $\frac{1}{2}\, \dot{w}c_p\Delta t(dT_i/dt)\Delta t$. These terms had been omitted from Eq. (2) and (2a) for sake of clarity. The complete equation for the heat balance in the volume is

$$T_b' = T_b + \Delta t/\mathrm{CAP}[\dot{q} + \dot{w}c_p\Delta T + \tfrac{1}{2}\dot{w}c_p(dT_i/dt - dT_0/dt)\Delta t] \quad (3)$$

For the full gradient, or for the steady-state condition, $T_b' = T_{av} + \frac{1}{2}(dT_i/dt + dT_0/dt)\Delta t$. From this the stability equation results:

$$(T_b - T_{av})\mathrm{CAP} + [\dot{q} + \dot{w}c_p\Delta T - (\mathrm{CAP}/2) \times (dT_i/dt + dT_0/dt)]\Delta t + \tfrac{1}{2}\dot{w}c_p(dT_i/dt - dT_0/dt)\Delta t^2 = 0 \quad (4)$$

In the example of the previous section, T_0 has risen from 100° to 180°F during 1 hr. Therefore, for the downstream node $dT_i/dt = 80$°F/hr. For $\Delta t_t = 1$ hr, substitution into Eq. (3) yields $T_b' = 116$°F. This means that a rise in energy stored in the volume took place. This is represented by the area of the triangle under the gradient. Using geometry it is found that the gradient which starts with $T_i = 180$°F, terminates 0.4 L inside the node. During this time $dT_0/dt = 0$ and will remain zero until the full gradient is reached.

Using these values ($T_i = 180$°F, $T_b = 116$°F, $dT_i/dt = 80$°F/hr), substitution into Eq. (4) yields a critical time step $\Delta t_H = 1.5$. At the end of this interval $T_i = 300$°F and, from Eq. (3), $T_b' = 200$°F. In other words, the gradient has completely shifted into node 2 and at this instant constitutes the full gradient through this node (see Fig. 2). It took 2.5 hr to develop the full gradient from the moment dT_i/dt started to rise in node 2, or 3.75 hr from the moment the gate was released.

At this point the rate of enthalpy carried in the stream is 2(200) = 400 Btu/hr. Considering the rise of T_i as a source of energy, the rate of supply is $\mathrm{CAP}(dT_i/dt) = 5 \times 80 = 400$ Btu/hr. This shows that an equilibrium exists between the rise of T_i and the enthalpy carried in the stream,

$$\mathrm{CAP}(dT_i/dt) = \dot{q} + \dot{w}c_p\Delta T \quad (5)$$

Under the conditions of the example a ΔT of 200°F must result from $dT_i/dt = 80$°F/hr.

Up to this instant dT_0/dt has remained zero. From now on T_0 must rise. To evaluate the rate of rise, three conditions must be examined, dT_i/dt becomes zero at the moment the full gradient is reached, dT_i/dt remains constant, or dT_i/dt assumes a new value. In the last condition the equilibrium at the inlet is upset, and the system has to develop a new gradient. Let T_i, T_b, and T_0 be the temperatures of the starting full gradient and T_i', T_b', T_0' the temperatures of the new gradient developed because of the new dT_i'/dt. The new equilibrium is stated by Eq. (5), $\mathrm{CAP}(dT_i'/dt) = \dot{q} + \dot{w}c_p(T_i' - T_0')$. Substituting $T_i' - T_0' = \Delta T + (dT_i/dt - dT_0/dt)\Delta t$ into this equation and transposing yields

$$\dot{w}c_p(dT_i/dt - dT_0/dt)\Delta t = \mathrm{CAP}(dT_i/dt) - \dot{q} - \dot{w}c_p\Delta T \quad (6)$$

Since in the case of a full gradient the first term of Eq. (4) is zero, substitution of the right half of Eq. (6) into the third term of Eq. (4) gives the change in T_0

$$dT_0/dt = (\dot{q} + \dot{w}c_p\Delta T)/\mathrm{CAP} \quad (7)$$

Equation (7) can always be used to evaluate dT_0/dt, as long as the first term of Eq. (4) is zero, that is, as long as a full gradient exists. It applies also when dT_i/dt becomes zero, since this term does not appear in Eq. (7). In the previous section the change in T_0 was derived by geometry; it could have been evaluated by Eq. (7).

Equation (7) also could be used when dT_i/dt remains constant. However, this would be a superfluous step. As long as dT_i/dt remains constant, ΔT must remain constant, as was shown above, and T_b and T_0 must rise at the same rate as does T_i.

Returning to the example, a full gradient has developed, $T_i = 300$°F, $T_b = 200$°F, and $T_0 = 100$°F. Up to this point,

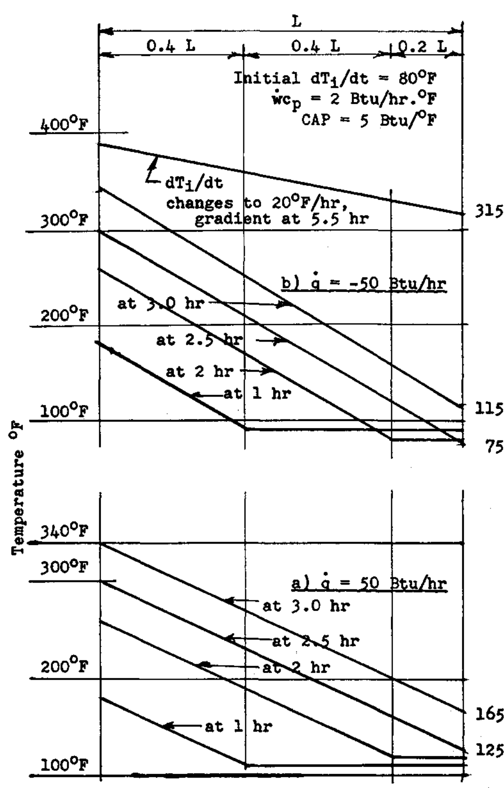

Fig. 3 Development of gradients when conditions are combined.

dT_i/dt was 80°F/hr, and now it changes to 20°F/hr. Equation (4) reveals that $\Delta t_H = 2.5$ hr; from Eq. (7), $dT_0/dt =$ 80°F/hr. Hence, after 2.5 hr (that is 6.25 hr after opening the gate) $T'_i = 300 + 2.5 \times 20 = 350$°F, $T'_0 = 100 + 2.5 \times 80 = 300$°F, and from Eq. (3), $T'_b = 325$°F. According to Eq. (5), $\Delta T = 50$°F is in equilibrium with $dT_i/dt = 20$°F/hr.

Returning to the full gradient at 3.75 hr problem time, but continuing $dT_i/dt = 80$°F/hr, Eq. (4) does not yield a critical Δt_H, since all terms are zero. This means that T_b and T_0 will rise at the same rate as does T_i as long as dT_i/dt remains constant. This can be verified by substitution into Eq. (7). Assuming $\Delta t = 1.5$ hr, at 5.25 hr problem time $T_i' = 300$°F $+ 1.5 \times 80 = 420$°F, $T_b' = 200$°F $+ 1.5 \times 80 = 320$°F, and $T_0' = 100$°F $+ 1.5 \times 80 = 220$°F; ΔT has remained 200°F. This condition is shown in Fig. 2 in dotted lines.

Combination of These Conditions

In the previous two sections, $\dot{q} = 0$ was assumed, and only the effect of enthalpy in the stream was explored. The combination of these conditions is shown in Fig. 3. Let us assume the same system, $T_i = T_b = T_0 = 100$°F, $\dot{w} = 2$ lb/hr, and CAP = 5 Btu/°F; further, $dT_i/dt = 80$°F/hr, and $\dot{q} = 50$ Btu/hr, hence, $dT_0/dt = 10$°F/hr. $\Delta t_t = 1$ hr, which satisfies Eq. (4). At the end of Δt, $T_i' = 180$°F, $T_0' = 110$°F, and from Eq. (3) $T_b' = 124$°F. T_b' can be verified by geometry. At the end of a second $\Delta t_t = 1$ hr $T_i'' = 260$°F, $T_0'' = 120$°F, and from Eq. (3) $T_b'' = 176$°F. Attempting a third time step $\Delta t_t = 1$, Eq. (4) reveals a critical $\Delta t_H = 0.5$ hr. At the end of Δt_H (that is at 2.5 hr problem time) $T_i''' = 300$°F, $T_0''' = 125$°F, and $T_b''' = 212.5$°F. Equilibrium with the inlet rise has been reached since Eq. (5) is satisfied, $80 \times 5 = 50 + 2 \times 175$. $\Delta T = 175$°F as long as $dT_i/dt = 80$°F/hr.

Now 0.5 hr remains to complete Δt_t. dT_i/dt remains constant and in accordance with the discussion of the previous section $dT_i/dt = dT_0/dt = dT_b/dt$. Hence all three rise through 40°F during the remaining 0.5 hr and at 3 hr problem time $T_i = 340$°F, $T_0 = 165$°F, and $T_b = 252.5$°F. This is shown in Fig. 3a.

Let us assume the same initial condition, but let $\dot{q} = -50$ Btu/hr. $dT_i/dt = 80$°F/hr and $\Delta t_t = 1$ hr. The particle that entered at the beginning of Δt at 100°F is cooled to 90°F while moving 0.4 L downstream. The particle entering at the end of Δt enters at 180°F. Hence $T_i' = 180$°F, $T_0' = 90$°F, and from Eq. (3) $T_b' = 108$°F. The profile is shown in Fig. 3b. At the end of a second Δt_t, $T_i'' = 260$°F, $T_0'' = 80$°F, and $T_b'' = 152$°F. Computing over a third $\Delta t_t = 1$ hr, a $\Delta t_H = 0.5$ hr is obtained from Eq. (4), hence the one hour interval must be subdivided. At 2.5 hr problem time, $T_i''' = 300$°F, $T_0''' = 75$°F, and $T_b''' = 187.5$°F. The full gradient has been reached which is in equilibrium with the inlet rise, since $\Delta T = 225$°F satisfies Eq. (5). During the remaining 0.5 hr the gradient must rise uniformly. At 3 hr problem time $T_i = 340$°F, $T_0 = 115$°F, and $T_b = 227.5$°F.

At this point dT_i/dt changes to 20°F/hr. The equilibrium has been upset. From Eq. (4) $\Delta t_H = 2.5$ hr is obtained and Eq. (6) yields $dT_0/dt = 80$°F/hr. At 5.5 hr problem time $T_i = 390$°F, $T_0 = 315$°F, and from Eq. (3) $T_b = 352.5$°F. The full gradient is in equilibrium since by Eq. (5) $5 \times 20 = -50 + 2 \times 75$. This condition is shown in Fig. 3b.

Had dT_i/dt been changed to 0°F/hr instead of to 20°F/hr, T_i would have remained at 340°F and, at 5.5 hr problem time, $T_0 = 315$°F and $T_b = 327.5$°F. If thereafter both $\dot{q} = -50$ Btu/hr and $T_i = 340$°F are maintained, no changes can take place. The node is at steady state, $\dot{q} + \dot{w}c_p\Delta T = 0$, hence Eq. (7) yields $dT_0/dt = 0$.

Summary and Example

The foregoing shows that it was possible to find dT_0/dt for all conditions examined, and therefore to compute T_b as a measure of energy stored in the volume. The heat-transfer rate $\dot{q}$ always determines dT_0/dt with the following exceptions: when a full gradient develops and dT_i/dt is in equilibrium with ΔT, $dT_0/dt = dT_i/dt$; when a full gradient develops but the equilibrium is upset, dT_0/dt is computed by Eq. (7); when enthalpy carried in causes T_b to rise above T_{av}, dT_0/dt, as computed from $\dot{q}$, must be modified in a manner discussed in the section "Enthalpy in Stream as Driver;" finally, when the system is at steady-state and $dT_i/dt = 0$, dT_0/dt must also be zero. The examples used were intended to be illustrative and to show the individual components which contribute to the energy balance in a fluid node.

In order to present a more realistic transient type of example, the following case was run with the computer program: flow at a rate of 62.5 lb/hr takes place in a 6-in.-diam pipe, 4.8 ft long. Initially the entire system is at 100°F. The temperature in the reservoir rises during the first 0.1 hr to 2000°F, maintains this temperature to 1.2 hr problem time and then drops to 200°F during the next 0.2 hr. In order to have a clearer view of the movement of the gradients, the flow rate has been kept constant and the tube is maintained at 100°F throughout the problem. The tube has been divided into 10 nodes 0.3 ft long plus 3 nodes 0.6 ft long; c_p of the fluid varies from 1.0 to 1.08. Figure 4 shows the gradients through the tube at various points in time. No attempt has been made to curve-fit the profiles, since the straight-line connection of outlet temperatures seemed a truer representation of the method of computation.

Fluids of Low Thermal Capacitance

Before considering thermal effects in the flow of such fluids, the concept of the "zero-capacitance node" within a thermal network must be explored. When the mass of a node is very small compared to that of the surrounding nodes, the heat stored in it is also very small compared to that stored in the larger nodes. Since a small node causes a small Δt_t it slows down computer runs and can therefore be expensive. Consequently, the mass of a node is set to zero when the error so introduced falls within the over-all accuracy of the method. It is hereby taken out of the time domain, since the Fourier equation can no longer be used. Instead, the node assumes a temperature which is the average of the surrounding temperatures, weighted by the resistor values of the connections. Thus, when a zero-capacitance node is connected to only one node, it assumes the temperature of that node.

The mass of a gas is usually very small compared to that of the duct and can therefore be set to zero. This means that the fluid assumes the temperature of the wall. However, a gradient must exist at the inlet if $T_i \neq T_w$.

Writing the heat balance equation, but replacing T_0 by T_w, further assuming a full gradient through the node, hence $T_b = (T_i + T_w)/2$

$$PL_rh_c(T_w - T_b) + \dot{w}c_p(T_i - T_w) = 0 \tag{8}$$

from which follows that,

$$L_r = 2\,\dot{w}c_p/Ph_c \tag{9}$$

where L_r is the reference node length corresponding to the full gradient. This means that only when the length of a node L is exactly L_r does $T_0 = T_w$ and at the same time $T_b = (T_i + T_w)/2$. However, the nodal length usually is not L_r. When $L < L_r$, T_0 has not yet reached T_w, and the gradient is shortened in direct proportion to L/L_r, and in Eq. (8) T_w must be replaced by this T_0, and $T_b = (T_i + T_0)/2$ must be used. If, on the other hand, $L > L_r$, heat-transfer takes place only over length L_r, whereafter $T_0 = T_w$, and $\dot{q} = 0$ in the remaining length of the node. If L_r, as computed by Eq. (9), is used, $T_b = (T_i + T_w)/2$ in Eq. (8).

If T_b, in a node longer than L_r, were computed by totalling the areas under the gradient, similar to the method shown in Fig. 1, $T_b \neq (T_i + T_w)/2$ would result. This T_b, when entered into Eq. (8), together with the actual node length L, would yield the same heat-transfer rate to the entire node, as when heat-transfer is considered to take place only over L_r and $T_b = (T_i + T_w)/2$.

An approximation is made and should be pointed out. The method assumes low regions of Reynolds number where the flow behaviour is essentially noncompressible. As flow rates increase, irregularities at the tube entrance and the mechanism of mixing become more important. Also mass storage in a node due to compressibility becomes significant. To maintain the mass balance, mass storage must change the flow rates, and this imposes stability criteria which are considerably more severe than those imposed by the thermal network. In this region the method described here cannot be used.

Considerations in Modelling

Correct evaluation of $\dot{q}$ depends on the use of the correct T_b in the Newton equation, and we have shown that $T_{av} = T_b$ only when a full gradient through the node or a steady-state exists. We also have shown that the time taken from one full gradient to the next or to a steady-state condition is exactly the time required for the entire mass in the volume to be replaced. [Equation (4) in the examples, when a full gradient existed, always resulted in Δt_H = 2.5 hr, which is CAP/$\dot{w}c_p$.]

If, therefore, the system can be modelled in such a way that Δt_t, as dictated by the thermal network, equals Δt_H, correct answers will result from the assumption that $T_b = T_{av}$. However, such a modelling effort is very time consuming. Also, the node size becomes very small with low flow rates,[4] and this could become expensive in computer time. Further, if thermal properties are temperature-dependent, Δt_t will vary throughout the problem, and this would introduce inaccuracies in the hydraulic computations, since the node sizes are fixed. If, in addition, $\dot{w}$ varies throughout the problem, results can become quite inaccurate.

The errors introduced when Δt_t differs from Δt_H and $T_b = T_{av}$ is assumed, follow directly from the foregoing examples. If in a fluid with thermal capacitance $\Delta t_t < \Delta t_H$, T_{av} cannot be used as T_b, as is shown in Fig. 1, but must be computed as described therein. If $\Delta t_t > \Delta t_H$, T_0 can rise only during Δt_H; it cannot rise during the remainder of the time step. If in a noncapacitance fluid $\Delta t_t < \Delta t_H$ (in this case Δt_H is the time needed for flow to take place through distance L_r), $T_0 \neq T_w$. If $\Delta t_t > \Delta t_H$, T_{av} cannot be used as T_b if the actual length L of the node is used in Eq. (7), however, $T_0 = T_w$.

Lack of recognition of these conditions is the reason for the frequently occurring oscillations of temperatures. These difficulties are removed by the method described herein, but certain limitations remain. When full gradients are developed by a steady state with $\dot{q}$ from the wall or by equilibrium with dT_i/dt, these gradients are stable, and their degree of accuracy is the same as is that of the over-all network.

In Fig. 2 gradients were developed by the sudden opening of a gate. Had the first node been selected one half as long, the gradient would have been twice as steep. Had the second node been longer than it was in the example, it would have taken longer for the full gradient to develop; that is, the gradient in this node would have been less steep and the rise of T_0 would have been delayed. In other words, gradi-

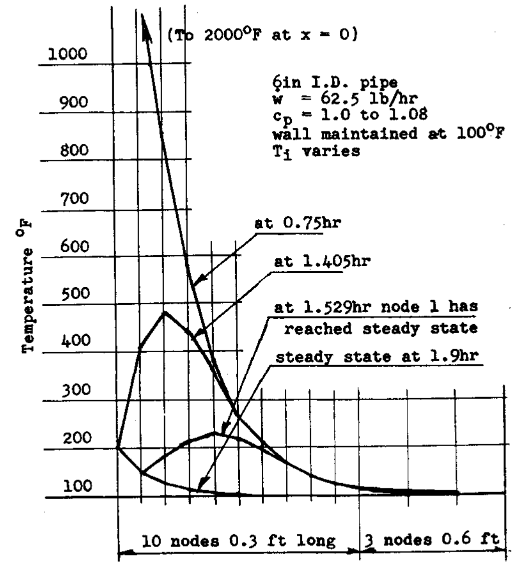

Fig. 4 Temperature profiles at several time points in response to variations in reservoir temperature.

ents caused by enthalpy in the stream depend on node size because of the assumption of linearity of the mixing process.

In the example the system is insulated from the environment, and conduction along the tube is neglected. In reality, conduction along the tube is present and has a damping effect. Still, this fact must be kept in mind when selecting node sizes.

If a finite time step were associated with the opening of the gate, the first node should be of such a length that one half of the fluid is replaced by fluid from the reservoir (which is at a different temperature) by the time the gate is fully opened.

The following computer run illustrates other conditions where selection of node size can be critical. The same system represented in Fig. 4 is used. The pipe is 6 ft long and is divided into 10 equal nodes. Initially the system is at 100°F. While the reservoir is held at this temperature (T_i remains 100°F throughout), the wall is instantaneously changed to 2000°F and held at 2000°F. (The sudden

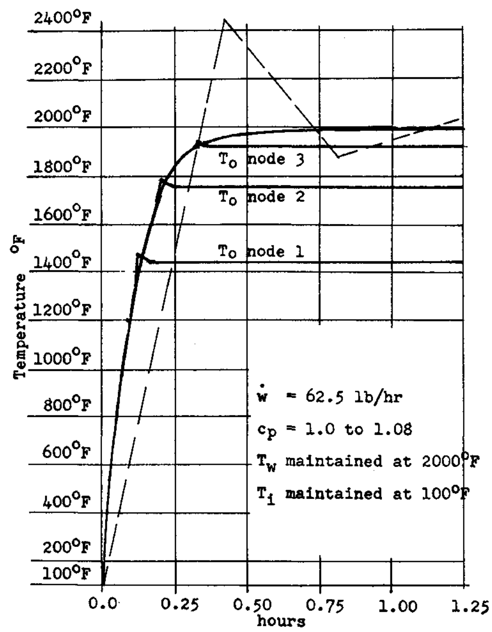

Fig. 5 Histories of outlet temperatures when inlet and wall temperatures are held constant.

change in wall temperature may seem unrealistic, but the problem is stated in this way to demonstrate better the development of the gradients.) Figure 5 shows the histories of rise of T_0's. When steady state is reached, this curve is also the profile of temperatures through the tube, if the abscissa were to represent nodal lengths. From this graph it appears that the second half of the tube could be expressed as one node. Another trial run was performed with six nodes: five of 0.6 ft each, and one of 3 ft. Results for the first 5 nodes were of course identical, and T_0 of the last node was only 4°F higher in the second case than it was in the 10-node system.

Another trial run, this time with three nodes 2.0 ft long, was performed. The resulting profile is shown by the dashed lines in Fig. 5. The over-all heat balance is maintained by this "curve;" however, three nodes clearly do not suffice to represent a curve which results from a temperature difference of 1900°F. While the severe stability criteria described in Ref. 1 are removed, a limitation on the node size still exists, particularly in the case of small $\dot{w}$'s and of large temperature rises; it is imposed by the assumption of linearity. Since it is not possible to give a simple formula for selecting node sizes, the manner in which the method breaks down has been demonstrated.

The small "over-shoots" of the T_0's in Fig. 5 occur because all parameters are assumed constant over the time step; hence, a node reaches steady state with the $\dot{q}$ value equal to that of the beginning of the time step. The energy balance quickly forces the temperature to its correct value. Correction by a backward differencing scheme has not been attempted. Gain in accuracy is questionable if compensation is made only to $\dot{q}$, since specific heat and viscosity are also variables. Further, radiation is frequently present. To increase the computer time by such a scheme did not seem justified, particularly since the basis for the computations of the parent program is a forward differencing technique.

In the normal temperature ranges overshooting is much less accentuated than it is in the ΔT of 1900°F of the example which was used to break down the method. Even there it falls within the accuracy limitations imposed by tolerances of material properties and manufacture and is self-correcting.

References

[1] Dusinberre, G. M., *Heat Transfer Calculations by Finite Differences*, International Textbook, Scranton, Pa., 1961, Chap. 1–3.

[2] Schneider, P. J., *Conduction Heat Transfer*, Addison-Wesley, Reading Mass., 1957, Chaps. 7 and 12.

[3] Kreith, F., *Principles of Heat Transfer*, International Textbook, Scranton, Pa., 1967, pp. 96–116, 175–188.

[4] French, R. J., Jr., "Analysis and Experimental Verification of Space Radiator Performance," Paper 845B, 1964, Society of Automotive Engineers—American Society of Civil Engineers, N.Y.